Math 255B Lecture 1 Notes

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1 Fredholm Theory

1.1 Fredholm operators

Definition 1.1. Let B_1, B_2 be Banach spaces. An operator $T \in \mathcal{L}(B_1, B_2)$ is called **Fredholm** if the kernel $\ker T = \{x \in B_1 : Tx = 0\}$ and the cokernel $\operatorname{coker} T = B_2 / \operatorname{im} T$ are finite-dimensional. We define the **index** if T to be $\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T \in \mathbb{Z}$.

Remark 1.1. If $T \in \mathcal{L}(B_1, B_2)$, then $\ker T$ is a closed subspace of B_1. However, $\operatorname{im} T$ need not necessarily be closed: take $B_1 = B_2 = C([0,1])$ and $(Tf)(x) = \int_0^x f(y)\, dy$.

So this is an algebraic condition. However, this implies an analytic condition on T:

Proposition 1.1. *If* $T \in \mathcal{L}(B_1, B_2)$ *and* $\dim \operatorname{coker} T < \infty$, *then* $\operatorname{im} T$ *is closed.*

Proof. We may assume T is injective, for otherwise, we can consider $\widetilde{T} : B_1 / \ker T \to B_2$ sending $x + \ker T \mapsto Tx$; then $\operatorname{im} \widetilde{T} = \operatorname{im} T$, and $\widetilde{T}$ is injective. Let $\dim \operatorname{coker} T = n < \infty$, and let $x_1, \dots, x_n \in B_2$ be such that $x_1 + \operatorname{im} T, \dots, x_n + \operatorname{im} T$ form a basis for $\operatorname{coker} T$. Let $S : \mathbb{C}^n \to B_2$ send $(a_1, \dots, a_n) \mapsto \sum_{j=1}^n a_j x_j$. Then S is injective, and $B_2 = \operatorname{im} T \oplus \operatorname{im} S$. It follows that $T_1 : B_1 \oplus \mathbb{C}^n \to B_2$ sending $(x, a) \mapsto Tx + Sa$ is a bijection. By the open mapping theorem, T_1 is a linear homeomorphism. Then $\operatorname{im} T = T_1(B_1 \oplus \{0\}) \subseteq B_2$ is closed. $\square$

1.2 Behavior of the index under perturbation

If $\dim B_j < \infty$ for $j = 1, 2$, then

$$\operatorname{ind} T = \dim \ker T - (\dim B_2 - \dim \operatorname{im} T) = \dim B_1 - \dim B_2.$$

Remarkably, for Fredholm operators, this property also extends to a similar property in the infinite dimensional case.

Theorem 1.1. *Let* $T \in \mathcal{L}(B_1, B_2)$ *be a Fredholm operator. If* $S \in \mathcal{L}(B_1, B_2)$ *is such that* $\|S\|_{\mathcal{L}(B_1, B_2)}$ *is sufficiently small, then* $T + S$ *is Fredholm, and* $\operatorname{ind}(T + S) = \operatorname{ind} T$.

To prove this, we have a lemma.

Lemma 1.1. *Let B be a Banach space, and let $S \in \mathcal{L}(B,B)$ be such that $\|S\| < 1$. Then $1 - S$ has an inverse (so* $\operatorname{ind}(1-S) = 0$*).*

Proof. The Neumann series $R = \sum_{k=0}^{\infty} S^k$ converges in $\mathcal{L}(B,B)$, and $R(1-S) = (1-S)R = 1$. $\square$

Remark 1.2. If $T \in \mathcal{L}(B_1, B_2)$ is invertible and $\|S\|$ is small, then $T + S$ is invertible: $T + S = T(1 + T^{-1}S)$ is invertible if $\|S\| < 1/\|T^{-1}\|$.

To prove the theorem, we will reduce to this case.

Proof. Write $n_+ = \dim \ker T$ and $n_- = \dim \operatorname{coker} T$. Let $R_- : \mathbb{C}^{n_-} \to B_2$ be injective and such that $B_2 = \operatorname{im} T \oplus R_-(\mathbb{C}^{n_-})$ (as we have constructed before). Let $e_1, \dots, e_{n_+}$ be a basis for $\ker T$, and let $\varphi_1, \dots, \varphi_{n_+} \in B_1^*$ be such that

$$\varphi_j(e_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

for all j, k; such continuous, linear forms exist by Hahn-Banach. Let $R_+ : B_1 \to \mathbb{C}^{n_+}$ send $x \mapsto (\varphi_1(x), \dots, \varphi_{n_+}(x))$. Then R_+ is surjective, and $R_+|_{\ker T}$ is bijective.

Let us introduce the **Grushin operator**[1]

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+}.$$

We claim that $\mathcal{P}$ is invertible: If $\mathcal{P}\begin{bmatrix} x \\ a_- \end{bmatrix} = 0$, then $Tx + R_-a_- = 0$ and $R_+x = 0$. Then $a_- = 0$, so $x \in \ker T$. Since R_+ is bijective on $\ker T$, we get $x = 0$. For surjectivity, we want to solve $Tx + R_-a_- = y$ and $R_+x = b$. Write $y = Tz + R_-c_-$. Then $a_- = c_-$ and $x - z \in \ker T$, so $x = z + \sum \alpha_j e_j$. We can take $\alpha_j = b_j - \varphi_j(z)$ for $1 \leq j \leq n_+$.

If $\|S\|$ is small enough, then

$$\widetilde{\mathcal{P}} = \begin{bmatrix} T + S & R_- \\ R_+ & 0 \end{bmatrix}$$

is invertible, and we introduce the inverse

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} : B_2 \oplus \mathbb{C}^{n_+} \to B_1 \oplus \mathbb{C}^{n_-}.$$

We will finish the proof next time. $\square$

[1] This terminology is not necessarily standard.